

Effective chiral Lagrangians for spin-1 mesons

Michael C. Birse

Theoretical Physics Group, Department of Physics and Astronomy,

University of Manchester, Manchester, M13 9PL, U.K.

Abstract

The commonly used types of effective theory for vector mesons are reviewed and their relationships clarified. They are shown to correspond to different choices of field for spin-1 particles and the rules for transforming between them are described. The importance of respecting chiral symmetry is stressed. The choice of fields that transform homogeneously under the nonlinear realisation of chiral symmetry imposes no preconceptions about the types of coupling for the mesons. This representation thus provides a convenient framework for relating different theories. It is also used to elucidate the nature of the assumptions in specific hidden-gauge and massive Yang-Mills models that have been widely used.

I. INTRODUCTION

At very low energies strong interactions among pions can be described by an effective Lagrangian based on a chirally symmetric sigma model [1]. To extend such a description to higher energies heavier mesons need to be incorporated, most notably vector mesons. Various schemes for doing so have been proposed, differing in the transformation properties of their vector fields under chiral symmetry.

Many of these approaches are motivated by the phenomenologically successful ideas of vector-meson dominance and universal coupling [2,3]. These lead to kinetic terms and couplings for the spin-1 mesons that have the same forms as in a gauge theory, reflecting the assumed universal coupling of these mesons to conserved currents. Examples include the “massive Yang-Mills” [4–6] and “hidden-gauge” theories [7]. In these approaches, low-energy theorems of chiral symmetry place important constraints on the gauge-type coupling of the ρ meson to two pions. It is essential that such Lagrangians respect chiral symmetry, otherwise they can lead to unrealistic results.

However it is not necessary to impose a gauge structure on the effective Lagrangian from the start. An alternative scheme for incorporating these mesons was suggested by Weinberg [8] and developed further by Callan, Coleman, Wess and Zumino [9]. In this treatment, denoted here by WCCWZ, the fields transform homogeneously under a nonlinear realisation of chiral symmetry. Another related approach is that of Ecker *et al.* in which the spin-1 mesons are represented by antisymmetric tensor, rather than vector, fields [10,11]. In contrast to the gauge-type theories, these formalisms have $\rho\pi\pi$ couplings that involve higher powers of momentum and are not directly constrained by chiral symmetry.

Despite the rather different forms of their Lagrangians, and the different types of coupling contained in them, all of these approaches are in principle equivalent. Each corresponds to a different choice of fields for the spin-1 mesons. This is illustrated rather well in extended Nambu–Jona-Lasinio models [12,13], where there is considerable freedom in the choice of auxiliary fields in the vector and axial channels. To some extent the choice of scheme must be based on the simplicity of the resulting Lagrangian. In making comparisons between the approaches it is important not to confuse features that arise from the choice of field with those that arise from requiring, for instance, universal coupling of the vector mesons. The former are not physical, controlling merely the off-shell behaviour of scattering amplitudes. The latter do have physical consequences, such as relations between on-shell amplitudes for different processes.

Effective Lagrangians of spin-1 mesons were extensively reviewed by Meissner in 1988 [6].

Since then there have been a number of developments in the field, most notably in connection with chiral perturbation theory [10,11]. Moreover interest in these Lagrangians has recently been reawakened by the possibility that experiments at high-luminosity accelerators such as CEBAF or DAΦNE may be able to explore some of the couplings that have up to now been inaccessible [14,15]. Another source of interest arises from the use of these theories in calculations of charge-symmetry violation in nuclear forces [16]. In addition, they are being used in studies of the behaviour of vector mesons in hot and dense matter [17–19] and some of them lead to quite different predictions for the mass of the ρ in matter [20]. In all of these contexts it is important to be able to compare theories, even though they may be expressed in different formalisms, in a way that is independent of the different choices of fields. To this end I explore here the connections between the various approaches, and their corresponding fields.

In Sec. II, I review some basic ideas of chiral symmetry, focussing on the constraints that the symmetry imposes on interactions with a vector character through the Weinberg-Tomozawa low-energy theorem [21,22] for pion scattering from a target with nonzero isospin and the KSFR relation [23] for the couplings of the ρ meson. The ingredients needed for construction of effective Lagrangians using the nonlinear realisation of chiral symmetry are also outlined.

The WCCWZ scheme [9] is introduced in Sec. III. I use it throughout this paper as an overall framework to compare and relate the other approaches since it imposes no prejudices about the forms of the couplings among the mesons. As noted by Ecker *et al.* [11], the consequences of physical assumptions like vector dominance can then be rather transparently expressed as relations between the couplings in a WCCWZ Lagrangian. By converting commonly used hidden-gauge and massive Yang-Mills theories into their WCCWZ equivalents, their couplings can be directly compared.

Within this scheme, the leading contributions to low-energy $\pi\pi$ scattering arise from four-pion interaction terms in the Lagrangian; ρ -exchange contributions are suppressed by powers of the pion momenta. The coupling constants for these four-pion interactions can

thus be determined from ChPT at order p^4 [24,11]. It turns out that their values are in good agreement with those obtained using the assumption of resonance saturation [24,10,11,25] in the corresponding channel of $\pi\pi$ scattering. (This assumption is related to, but not as strong as, that of vector dominance, as discussed in Sec. III.) Moreover these four-point interactions are essential if the effective theory is to be well-behaved at short distances. For example if the Hamiltonian is to be bounded from below, the four-point couplings should satisfy inequalities relating them to three-point ones [26,27]. The corresponding equalities are then obtained from the stronger assumption of resonance saturation by a single meson [11].

In the hidden-gauge approach [7], described in Sec. IV, an artificial local symmetry is introduced into the nonlinear sigma model by the choice of field variables. The ρ meson is then introduced as a gauge boson for this symmetry. As stressed by Georgi [28], the additional local symmetry has no physics associated with it, and it can be removed by fixing the gauge. In the unitary gauge the symmetry reduces to a nonlinear realisation of chiral symmetry, under which the vector fields transform inhomogeneously, in contrast to those of WCCWZ. However, with a further change of variable any vector-meson Lagrangian of the hidden-gauge form can be converted into an equivalent WCCWZ one [28]. The rules for transforming a Lagrangian from hidden-gauge to WCCWZ form have also been noted by Ecker *et al.* [11].

As I show here, by changing variables from the hidden-gauge to WCCWZ scheme, the gauge coupling constant of the former is really a parameter in the choice of vector field. This coupling constant does not appear in the equivalent WCCWZ Lagrangian and so hidden-gauge theories with different gauge couplings, together with different higher-order couplings, can be equivalent. The conventional choice is shown to be one that eliminates any $\mathcal{O}(p^3)$ $\rho\pi\pi$ coupling from the hidden-gauge Lagrangian, so that the leading corrections to the $\mathcal{O}(p)$ coupling are of order p^5 . If the $\gamma\rho$ mixing strengths satisfy a particular relation [11], then this choice of field also eliminates the leading momentum-dependent corrections, of order p^2 , to the mixing. This reduction of the momentum dependence of the couplings thus allows the

hidden-gauge approach to embody the empirical observation that the KSFR relation [23] is well satisfied by the $\rho\pi\pi$ and $\gamma\rho$ couplings determined from the decay of on-shell ρ mesons.

In massive Yang-Mills theories [4,6], described in Sec. V, the vector and axial fields transform under a linear realisation of chiral symmetry. Three- and four-point couplings among these fields are included and, together with the kinetic terms, form a Yang-Mills Lagrangian with a local chiral symmetry. The full theory does not possess this gauge symmetry since it includes mass terms which have only global symmetry. By changing variables to spin-1 fields that transform under the nonlinear realisation of chiral symmetry, any massive Yang-Mills theory can be converted into an equivalent WCCWZ one and its relations to other theories, such as hidden-gauge ones, can be explored.

The use of a linear realisation of chiral symmetry means that both the ρ meson and its chiral partner the a_1 must be treated on the same footing. One cannot simply omit the a_1 from a massive Yang-Mills theory without violating chiral symmetry. Nonetheless it is possible to write down Lagrangians with a Yang-Mills form for the ρ meson alone, provided that one takes care to include additional terms that ensure satisfaction of the chiral low-energy theorems [31]. As described here a convenient way to generate these terms is to take a hidden-gauge theory and make a change of variables that brings it into a Yang-Mills-like form.

The final formalism I consider is the one based on antisymmetric tensor fields [10,11,32] described in Sec. VI. These fields transform homogeneously under the nonlinear realisation of chiral symmetry and so the approach has many similarities with that of WCCWZ. The main difference is that the basic $\rho\pi\pi$ and $\gamma\rho$ couplings involve one less power of momentum. This means that, if resonance saturation is assumed, the Lagrangian can take a particularly simple form. For every coupling in the general WCCWZ Lagrangian, one can construct a corresponding one involving tensor fields [11,33] A more direct way using path integrals to translate between the two schemes has been described in [34].

In Sec. VII, I discuss briefly the explicit symmetry-breaking terms that can appear in these Lagrangians and comment on their applications to isospin-violating processes.

II. CHIRAL SYMMETRY

A. PCAC

The current masses of up and down quarks are very much smaller than typical hadron energy scales, and hence to a good approximation QCD is invariant under both ordinary isospin rotations,

$$\psi \rightarrow \left(1 - \frac{1}{2}i\boldsymbol{\beta} \cdot \boldsymbol{\tau}\right) \psi, \quad (2.1)$$

and axial isospin rotations,

$$\psi \rightarrow \left(1 - \frac{1}{2}i\boldsymbol{\alpha} \cdot \boldsymbol{\tau}\gamma_5\right) \psi, \quad (2.2)$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ denote infinitesimal parameters. Together these form the chiral symmetry group $SU(2)_R \times SU(2)_L$ [1]. (I concentrate here on the up and down quarks; the extension to three light flavours is straightforward.) The corresponding Noether currents are the (vector) isospin currents

$$\mathbf{J}^\mu = \bar{\psi}\gamma^\mu \frac{1}{2}\boldsymbol{\tau}\psi, \quad (2.3)$$

and the axial currents

$$\mathbf{J}_5^\mu = \bar{\psi}\gamma^\mu \gamma_5 \frac{1}{2}\boldsymbol{\tau}\psi. \quad (2.4)$$

The presence of small current masses for the up and down quarks means that this symmetry is only approximate. The axial currents thus have divergences proportional to these masses. Furthermore the difference between the up and down quark masses breaks isospin symmetry and so the vector currents have nonzero divergences.

Despite the smallness of the current masses, the QCD vacuum is not even approximately invariant under axial isospin rotations. The chiral symmetry is hidden (or “spontaneously broken”) and so degenerate states of opposite parity do not appear in the hadron spectrum. Instead the pions are close to being the corresponding massless Goldstone bosons. The hidden symmetry shows up in the forms of the interactions of low-energy pions.

One important consequence of this hidden symmetry is the nonzero matrix element for the weak decay of charged pions

$$\langle 0 | J_5^{a\mu}(x) | \pi_b(q) \rangle = i f_\pi q^\mu e^{-iq \cdot x} \delta_{ab}, \quad (2.5)$$

where the pion decay constant is $f_\pi = 92.4 \pm 0.3$ MeV [36]. The divergence of this equation is

$$\langle 0 | \partial_\mu J_5^{a\mu}(x) | \pi_b(q) \rangle = f_\pi m_\pi^2 e^{-iq \cdot x} \delta_{ab}, \quad (2.6)$$

which shows that the operators

$$\phi(x) = \partial_\mu \mathbf{J}_5^\mu(x) / (f_\pi m_\pi^2) \quad (2.7)$$

connect the vacuum and one-pion states with the same normalisation that canonical pion fields would have. These interpolating fields provide the basis for an approach known as “partial conservation of the axial current” (PCAC). This is a method for elucidating the consequences of approximate chiral symmetry for the interactions of low-energy pions.

Of course the use of these particular interpolating fields is a matter of choice. Other pion fields should give the same results for all physical amplitudes involving on-shell pions; where they differ is in their off-shell extrapolations. The advantage of the PCAC choice is that, in the soft-pion limit, we can relate amplitudes for interactions with pions to the axial transformation properties of the states involved.

For the present discussion of vector mesons, it is worth re-examining two of the consequences of PCAC. First consider the scattering of pions off a target with nonzero isospin. By applying LSZ reduction to the amplitude for forward scattering of a pion with momentum k off a target with momentum p , it can be written in the form (see, for example, [37])

$$F^{ab} = i \left(\frac{m_\pi^2 - k^2}{f_\pi m_\pi^2} \right)^2 \int d^4x e^{ik \cdot x} \langle p | T \left(\partial_\mu J_5^{a\mu}(x), \partial_\nu J_5^{b\nu}(x) \right) | p \rangle, \quad (2.8)$$

where I have used the PCAC pion fields defined by (2.7). On integrating by parts, the integral gives rise to two terms. One of these is an equal-time commutator which reduces to an explicit chiral symmetry-breaking matrix element (or “sigma commutator”) in the

soft-pion limit [21]. This piece is of order m_π^2 . The second term, which has a factor of k_μ , can be integrated by parts again to give two terms: another equal time commutator and term with a factor of $k_\mu k_\nu$. The first of these contains a piece that is linear in the pion energy and so forms the leading term in a chiral expansion of the amplitude.

This leading term can be found by taking the soft-pion limit, that is setting the pion three-momentum to zero and letting the energy tend to zero. The amplitude (2.8) (integrated by parts as just described) then reduces to

$$F^{ab} = \frac{1}{f_\pi^2} k^0 \langle p | [J_5^{a0}(0), Q_5^b] | p \rangle + \mathcal{O}(m_\pi^2, k^2), \quad (2.9)$$

where Q_5^a is the axial charge corresponding to (2.4). Using the algebra associated with the $SU(2)_R \times SU(2)_L$ symmetry group, the commutator in this expression is just

$$[J_5^{a0}(0), Q_5^b] = i\epsilon^{abc} J^{c0}(0). \quad (2.10)$$

Hence the leading term in the scattering amplitude has the form

$$F^{ab} = \frac{i}{f_\pi^2} k^0 \epsilon^{abc} \langle p | J^{c0} | p \rangle + \mathcal{O}(m_\pi^2, k^2). \quad (2.11)$$

Since $-i\epsilon^{abc}$ are just the matrix elements of the pion isospin operator, we can see that this amplitude is proportional to the scalar product of the isospin operators for the pion and target. This is the famous result derived by Weinberg and Tomozawa for low-energy pion-nucleon scattering [21,22]. The argument here shows that this form is general and should be present for pion scattering from any target.

The form of the Weinberg-Tomozawa term is exactly the same as one would get from exchange of a ρ meson coupled to the isospin currents of the pion and target. However it is important to remember that it arises as a consequence of chiral symmetry and has no necessary connection with ρ exchange. Moreover its strength is fixed by the symmetry alone. Hence in any approach where ρ meson is coupled to the isospin currents, as in a gauge theory, the requirement that ρ exchange does not violate this low-energy theorem places constraints on the couplings of the ρ to other particles.

A related result can be obtained from the amplitude for ρ decay into two pions. Applying an LSZ reduction to this as above, it can be written as

$$G^{ab} = i \left(\frac{m_\pi^2 - k^2}{f_\pi m_\pi^2} \right)^2 \int d^4x e^{ik \cdot x} \langle 0 | T \left(\partial_\mu J_5^{a\mu}(x), \partial_\nu J_5^{b\nu}(x) \right) | \rho(p) \rangle. \quad (2.12)$$

In the real world where the ρ is massive, there is no way to extrapolate this amplitude to the soft-pion limit of vanishing four-momenta. Nonetheless, at least in the context of an effective theory where the mass of the ρ appears as a parameter in the Lagrangian, one can ask how this amplitude should behave as the ρ mass is taken by hand to zero.

After manipulating this amplitude in the same way as for the pion scattering above, it can be written in the form

$$G^{ab} = \frac{i}{f_\pi^2} k^0 \epsilon^{abc} \langle 0 | J^{c0} | \rho(p) \rangle + \mathcal{O}(m_\pi^2 k, k^3), \quad (2.13)$$

in this artificial soft-pion, light- ρ limit. This relation links the $\rho\pi\pi$ coupling at first order in k to the ρ -to-vacuum matrix element of the vector current. The vector, isovector nature of the ρ means that there is no term analogous to the sigma commutator and so higher-order terms start at third order in k and m_π . The matrix element in (2.13) is responsible for the electromagnetic decay $\rho^0 \rightarrow e^+ e^-$ and can be expressed in the form

$$\langle 0 | J^{c0} | \rho(p) \rangle = g_{\rho\gamma}(m_\rho). \quad (2.14)$$

In the soft-pion limit, the only contribution to $\rho \rightarrow \pi\pi$ of order k arises from the coupling of the ρ to the pionic isospin current

$$\mathcal{L}_{\rho\pi\pi} = g_{\rho\pi\pi} \boldsymbol{\rho}^\mu \cdot \boldsymbol{\pi} \wedge \partial_\mu \boldsymbol{\pi}, \quad (2.15)$$

if such a term is present in the effective Lagrangian. Comparing the corresponding amplitude G^{ab} with (2.13) gives

$$g_{\rho\gamma}(0) = 2g_{\rho\pi\pi} f_\pi^2, \quad (2.16)$$

where both ρ couplings are evaluated in the soft-pion, light- ρ limit. This is the celebrated KSFR relation [23] in its first form. For any chirally-symmetric effective Lagrangian it relates the strength of $\gamma\rho$ mixing at zero four-momentum to that of the coupling in (2.15).

Although the KSFR relation has been derived by taking an unphysical limit where the ρ mass tends to zero, one can ask how close the real world is to this limit. Rather remarkably in view of the large ρ mass, if one assumes that the $\gamma\rho$ mixing is independent of momentum and the $\rho\pi\pi$ coupling is purely of the form (2.15), then the coupling strengths deduced from the observed ρ decays satisfy (2.16) to about 10%. This suggests that it should be possible to find a realistic effective Lagrangian for $\pi\rho$ physics in which higher-order corrections to these couplings are small. As we shall see in Sec. IV, the hidden-gauge scheme can provide just such a Lagrangian.

If one further assumes complete vector dominance in the photon-pion coupling [2,3], as described in more detail in Sec. III, then the ρ couplings also satisfy

$$g_{\rho\gamma} = \frac{m_\rho^2}{g_{\rho\pi\pi}}. \quad (2.17)$$

This can be combined with (2.16) to obtain a second form of the KSFR relation,

$$m_\rho^2 = 2g_{\rho\pi\pi}^2 f_\pi^2, \quad (2.18)$$

which is also satisfied remarkably well by the coupling deduced from ρ decay. If one assumes vector dominance in the couplings of the photon to all hadrons, which demands a universal coupling of the ρ to the isospin current, then this relation has a further interesting consequence: ρ exchange between a pion and another hadron, such as a nucleon, can account for the whole strength of the Weinberg-Tomozawa term discussed above.

B. Effective Lagrangians

The PCAC pion field is useful in studying processes that involve external pions. However it is not a canonical pion field (except in certain rather simple models) and so cannot be used to calculate the effects of virtual pions, either as exchanged particles or in loop diagrams. For these it is most convenient to work with effective field theories that embody the constraints

imposed by chiral symmetry on the couplings between the fields.¹

The simplest way to construct a Lagrangian satisfying chiral symmetry is to introduce fields that transform linearly under the group $SU(2)_R \times SU(2)_L$, in the same way that the corresponding quark bilinears do. In particular, pion fields $\phi(x)$ can be introduced along with a scalar field $\sigma(x)$ to form a multiplet (σ, ϕ) that transforms like $(\bar{\psi}\psi, \bar{\psi}i\gamma_5\tau\psi)$. The excitations of these fields are pions, which are almost massless, approximate Goldstone bosons, together with massive scalar mesons. If we are only interested in low-energy physics then we may eliminate the massive degrees of freedom by restricting these fields to the chiral circle, $\sigma(x)^2 + \phi(x)^2 = f_\pi^2$.

The Lagrangian for the resulting nonlinear sigma model can be expressed in terms of the $SU(2)$ matrix $U(x)$ defined by $f_\pi U(x) = \sigma(x) + i\tau \cdot \phi(x)$. Under the global $SU(2) \times SU(2)$ chiral symmetry this transforms as

$$U(x) \rightarrow g_L U(x) g_R^\dagger, \quad (2.19)$$

with $g_L, g_R \in SU(2)$. The matrix field $U(x)$ can be parametrised in terms of three pion fields in a variety of ways. Here I shall use the exponential form $U(x) = \exp(i\tau \cdot \pi(x)/f_\pi)$.

In the chiral limit, the leading piece of the Lagrangian is

$$\mathcal{L} = \frac{f_\pi^2}{4} \langle \partial_\mu U \partial^\mu U \rangle, \quad (2.20)$$

where $\langle \dots \rangle$ denotes a trace in $SU(2)$ space. This should be supplemented with terms describing explicit chiral symmetry breaking and interactions of higher-order in the pion momenta [24,1]. Estimates of the coefficients of these interactions based on the idea of resonance saturation [24,10,25] agree well with phenomenologically determined values. This has led people to consider extended Lagrangians that include fields describing the heavier particles whose exchanges can generate the higher-order interactions. Amongst these Lagrangians are the ones for spin-1 mesons discussed here.

¹A modern review of these ideas can be found in the book by Donoghue, Golowich and Holstein [1].

The most straightforward way to introduce the spin-1 ρ and a_1 mesons is to use fields $\widetilde{\mathbf{V}}^\mu$ and $\widetilde{\mathbf{A}}^\mu$ that transform like the vector and axial currents (2.3,4).² The transformation properties of these are most easily given in terms of their right- and left-handed combinations

$$\widetilde{\mathbf{X}}^\mu = \frac{1}{\sqrt{2}} (\widetilde{\mathbf{V}}^\mu + \widetilde{\mathbf{A}}^\mu), \quad \widetilde{\mathbf{Y}}^\mu = \frac{1}{\sqrt{2}} (\widetilde{\mathbf{V}}^\mu - \widetilde{\mathbf{A}}^\mu). \quad (2.21)$$

Under the chiral rotation above, these fields become

$$\begin{aligned} \widetilde{X}_\mu &\rightarrow g_R \widetilde{X}_\mu g_R^\dagger \\ \widetilde{Y}_\mu &\rightarrow g_L \widetilde{Y}_\mu g_L^\dagger, \end{aligned} \quad (2.22)$$

where I have introduced matrix fields, $X_\mu = \frac{1}{2} \boldsymbol{\tau} \cdot \mathbf{X}_\mu$, and so on. Such linearly transforming fields are the basis for the massive Yang-Mills theories described in Sec. V.

C. Nonlinear realisation

Although the transformation properties of the fields in (2.22) are simple, the couplings of these fields to pions do not necessarily vanish in the soft-pion limit. As a result, calculations of scattering amplitudes involve large contributions with strong cancellations between them which are needed to satisfy chiral low-energy theorems. For many purposes it would be simpler if the correct low-energy behaviour was already built into the interaction terms appearing in the Lagrangian. This can be achieved by switching to the nonlinear realisation of chiral symmetry introduced by Weinberg [8], which forms the basis for both the WC-CWZ and hidden-gauge schemes. This realisation of the symmetry is obtained from the transformation properties of the square root of $U(x)$, denoted by $u(x)$:

$$u \rightarrow g_L u h^\dagger(u, g_L, g_R) = h(u, g_L, g_R) u g_R^\dagger, \quad (2.23)$$

²These fields cannot however be simply identified with the corresponding currents since, like the PCAC interpolating pion fields, these currents are not canonical field operators.

where $h(u(x), g_L, g_R)$ is a compensating $SU(2)$ rotation which depends on the pion fields at x as well as on $g_{L,R}$. The detailed form of h is not needed here; it can be found in Ref. [9].

It is convenient to introduce the following field gradients

$$\begin{aligned} u_\mu &= i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) \\ \Gamma_\mu &= \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger). \end{aligned} \quad (2.24)$$

If these are expanded in powers of the pion field $\boldsymbol{\pi}(x)$, we find that the leading terms are

$$\begin{aligned} u_\mu &= -\frac{1}{f_\pi} \boldsymbol{\tau} \cdot \partial_\mu \boldsymbol{\pi} + \dots, \\ \Gamma_\mu &= \frac{i}{4f_\pi^2} \boldsymbol{\tau} \cdot \boldsymbol{\pi} \wedge \partial_\mu \boldsymbol{\pi} + \dots, \end{aligned} \quad (2.25)$$

from which we can see that they are an axial vector and vector respectively. Under the nonlinear realisation of chiral symmetry the transformation of u_μ is homogeneous

$$u_\mu \rightarrow h u_\mu h^\dagger, \quad (2.26)$$

whereas that of Γ_μ is inhomogeneous,

$$\Gamma_\mu \rightarrow h \Gamma_\mu h^\dagger + h \partial_\mu h^\dagger. \quad (2.27)$$

The simple sigma model (2.20) can be expressed in terms of u^μ as

$$\mathcal{L} = \frac{f_\pi^2}{4} \langle u_\mu u^\mu \rangle, \quad (2.28)$$

which from (2.26) is chirally invariant.

In fact Γ_μ is the connection on the coset space $SU(2) \times SU(2) / SU(2)$ and with it we can construct the covariant derivative on this space:

$$\nabla_\mu = \partial_\mu + [\Gamma_\mu, \cdot]. \quad (2.29)$$

The covariant derivatives of u_μ satisfy the useful relation

$$\nabla_\mu u_\nu - \nabla_\nu u_\mu = 0. \quad (2.30)$$

Also, the curvature tensor corresponding to Γ_μ can be expressed in terms of u_μ as

$$\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] = \frac{1}{4}[u_\mu, u_\nu]. \quad (2.31)$$

Vector and axial fields can be defined that transform homogeneously under this realisation symmetry, like u^μ ,

$$\begin{aligned} V_\mu &\rightarrow h V_\mu h^\dagger \\ A_\mu &\rightarrow h A_\mu h^\dagger. \end{aligned} \quad (2.32)$$

Such fields can be obtained from the linearly transforming ones described in (2.21) above by multiplying them by $u(x)$ and its conjugate:

$$\begin{aligned} X_\mu &= u \widetilde{X}_\mu u^\dagger \\ Y_\mu &= u^\dagger \widetilde{Y}_\mu u. \end{aligned} \quad (2.33)$$

These form the basis of the WCCWZ scheme described in the next section.

D. External fields

To describe electromagnetic and weak interactions, we also need to couple our hadron fields to external vector and axial fields. In the case of linearly realised chiral symmetry this is straightforward. The minimal couplings are obtained by replacing all derivatives with the corresponding gauge-covariant derivatives, for example:

$$\partial_\mu U \rightarrow D_\mu U = \partial_\mu U + i[U, v_\mu] + i[U, a_\mu]_+, \quad (2.34)$$

where v_μ and a_μ are the external fields. (Factors of coupling constants such as e have been absorbed into the definitions of these fields.)

Other, nonminimal couplings, such as anomalous magnetic moments for the vector mesons, can be included by adding terms to the Lagrangian that involve the gauge-invariant

field strengths $v_{\mu\nu}$ and $a_{\mu\nu}$. There should also be direct $\gamma\rho$ mixing. This can be included in a gauge-invariant manner through a term of the form

$$\mathcal{L}_{\rho\gamma(k)} = -\frac{g_{\rho\gamma}}{m_\rho^2} \langle V^{\mu\nu} v_{\mu\nu} \rangle, \quad (2.35)$$

where the value of $g_{\rho\gamma}$ is that for an on-shell ρ -meson. By making appropriate changes of variable such a Lagrangian can also be rewritten in a form where the mixing arises from a mass-type term [2,3,38]

$$\mathcal{L}_{\rho\gamma(m)} = -2g_{\rho\gamma} \langle V^\mu v_\mu \rangle, \quad (2.36)$$

where the gauge invariance is no longer transparent. If vector dominance (2.17) holds exactly, then this transformed Lagrangian contains no direct $\gamma\pi\pi$ coupling at first order in the pion momentum: the entire photon coupling to the pion arises from a virtual ρ .

The couplings discussed so far are those of isovector external fields. Electromagnetic interactions also contain isoscalar pieces, which are rather different in character. The leading isoscalar coupling of the photon is to three pions and this can be described by a term coupling the photon to the topological current

$$B^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\kappa\lambda} \langle (U^\dagger \partial_\nu U) (U^\dagger \partial_\kappa U) (U^\dagger \partial_\lambda U) \rangle, \quad (2.37)$$

which provides the baryon number current in the Skyrme model [39,40]. This coupling forms part of the anomalous Lagrangian, which also describes processes such as $\pi^0 \rightarrow \gamma\gamma$. Unlike the ones discussed so far, these terms are odd under the operation $U(x) \rightarrow U(x)^\dagger$ [41]. In the case of an effective theory with SU(3) symmetry, the $\gamma\pi^+\pi^0\pi^-$ coupling can be obtained from the gauged version of the Wess-Zumino-Witten term [42,41] (see also: [43,6]), although for an SU(2) theory it has to be added by hand.

With the nonlinear realisation of the symmetry, the use of gauge-covariant derivatives means that the definitions of the gradients (2.24) should be replaced by

$$\begin{aligned} u_\mu &= iu^\dagger [\partial_\mu - i(v_\mu - a_\mu)]u - iu [\partial_\mu - i(v_\mu + a_\mu)]u^\dagger \\ \Gamma_\mu &= \frac{1}{2} \left(u^\dagger [\partial_\mu - i(v_\mu - a_\mu)]u + u [\partial_\mu - i(v_\mu + a_\mu)]u^\dagger \right). \end{aligned} \quad (2.38)$$

In the nonminimal couplings, the field-strength tensors should appear in the nonlinearly transforming combinations,

$$F_{\pm}^{\mu\nu} = \frac{1}{2} \left(u(v_{\mu\nu} + a_{\mu\nu})u^{\dagger} \pm u^{\dagger}(v_{\mu\nu} - a_{\mu\nu})u \right), \quad (2.39)$$

where $F_{+}^{\mu\nu}$ is the combination that corresponds to a vector field coupling and $F_{-}^{\mu\nu}$ to an axial one. The tensor $F_{+}^{\mu\nu}$ also appears as an additional term in the curvature tensor defined using the covariant derivatives on the coset space:

$$\partial_{\mu}\Gamma_{\nu} - \partial_{\nu}\Gamma_{\mu} + [\Gamma_{\mu}, \Gamma_{\nu}] = \frac{1}{4}[u_{\mu}, u_{\nu}] - iF_{+\mu\nu}. \quad (2.40)$$

Explicit symmetry breaking arising from the current quark masses can be introduced by treating those masses as if they were uniform external scalar fields. In the nonlinear realisation, the corresponding terms in the Lagrangian can be expressed in terms of the quantities

$$\chi_{\pm} = u\mathcal{M}u \pm u^{\dagger}\mathcal{M}u^{\dagger}, \quad (2.41)$$

where \mathcal{M} is proportional to the matrix of current quark masses.

III. WCCWZ

In the WCCWZ approach [8,9,11] vector and axial fields transform homogeneously under the nonlinear realisation of chiral symmetry just described. This scheme imposes no constraints on the couplings of the spin-1 mesons, apart from those that follow from approximate chiral symmetry.

Denoting the fields describing the ρ and a_1 mesons by V^{μ} and A^{μ} and defining their covariant derivatives by

$$V_{\mu\nu} = \nabla_{\mu}V_{\nu} - \nabla_{\nu}V_{\mu}, \quad A_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu}, \quad (3.1)$$

we can then express the kinetic and mass terms of the Lagrangian as

$$\mathcal{L} = -\frac{1}{2}\langle V_{\mu\nu}V^{\mu\nu} \rangle - \frac{1}{2}\langle A_{\mu\nu}A^{\mu\nu} \rangle + m_V^2\langle V_{\mu}V^{\mu} \rangle + m_A^2\langle A_{\mu}A^{\mu} \rangle. \quad (3.2)$$

Expanding the covariant derivatives here to leading order in the pion fields, we find that this contains the term

$$-4\langle\partial_\mu V_\nu[\Gamma^\mu, V^\nu]\rangle = \frac{1}{f_\pi^2}(\mathbf{V}^\nu \wedge \partial_\mu \mathbf{V}_\nu) \cdot (\boldsymbol{\pi} \wedge \partial^\mu \boldsymbol{\pi}) \quad (3.3)$$

which corresponds to a local coupling between the isospin currents of the pions and ρ mesons. In low-energy $\pi\rho$ scattering, this is just what is needed to give the Weinberg-Tomozawa term (2.11). Note that in the WCCWZ scheme this piece is contained within the kinetic term of the Lagrangian and is generated without ρ -exchange contributions. Moreover all other contributions to $\pi\rho$ scattering arise from vertices with at least two factors of u_μ (to satisfy the restrictions of parity and isospin) and so are suppressed by at least one further power of the pion momentum.³

A general chirally symmetric Lagrangian for $\pi\rho a_1$ physics consists of all terms that can be constructed out of traces of products of u_μ , V_μ , A_μ and their covariant derivatives, and that are symmetric under parity. The derivatives can include both the antisymmetric ones of (3.1) and the symmetric combinations, as recently pointed out by Borasoy and Meissner [33]. Up to fourth-order in pion-field gradients and the vector fields, the WCCWZ Lagrangian includes the terms

$$\begin{aligned} \mathcal{L} = & \frac{f_\pi^2}{4}\langle u_\mu u^\mu \rangle - \frac{1}{2}\langle V_{\mu\nu} V^{\mu\nu} \rangle + m_V^2 \langle V_\mu V^\mu \rangle - \frac{i}{2}g_1 \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle + \frac{i}{2}g_2 \langle V_{\mu\nu} [V^\mu, V^\nu] \rangle \\ & + \frac{1}{8}g_3 \langle [u_\mu, u_\nu]^2 \rangle - \frac{1}{4}g_4 \langle [u_\mu, u_\nu] [V^\mu, V^\nu] \rangle + \frac{1}{8}g_5 \langle [V_\mu, V_\nu]^2 \rangle + \dots \end{aligned} \quad (3.4)$$

Even to this order this expression is not complete, but the terms written out explicitly here are those we shall need below in discussing the connection to the hidden-gauge Lagrangian of Bando *et al.* [35]. These include the famous $\langle [u_\mu, u_\nu]^2 \rangle$ term introduced by Skyrme to stabilise solitons in a nonlinear sigma model [39]. Obviously many other three- and four-

³If explicit chiral symmetry breaking terms are included then these also contribute at next-to-leading order in the chiral expansion, since they are proportional to m_π^2 .

point interactions, involving the axial as well as the vector field, should also be present in the full effective Lagrangian.

The leading $\rho\pi\pi$ coupling term in this scheme is

$$-\frac{i}{2}g_1\langle V_{\mu\nu}[u^\mu, u^\nu]\rangle = g_1\partial^\mu\mathbf{V}^\nu \cdot \partial_\mu\boldsymbol{\pi} \wedge \partial_\nu\boldsymbol{\pi}, \quad (3.5)$$

which is of order p^3 . Hence, as just mentioned, ρ exchange does not contribute to the Weinberg-Tomozawa term in $\pi\rho$ scattering. It also means that the first contribution of ρ exchange to low-energy $\pi\pi$ scattering is of order p^6 , as noted by Ecker *et al.* [11]. The leading contribution to $\pi\pi$ scattering is thus the Skyrme term, which is of order p^4 , and the corresponding coupling constant can be taken from analyses of $\pi\pi$ scattering using ChPT to that order [24,10,25]. The values of this and other four-pion interactions are in good agreement with those obtained by assuming resonance saturation.

A general Lagrangian in this scheme can suffer from unphysical behaviour at short distances unless its coupling constants satisfy certain relations [11,26,27]. A particularly transparent way to see this is to consider the corresponding Hamiltonian, as pointed out by Kalafatis [26]. In the presence of certain configurations of mean fields, the Hamiltonian for high-momentum modes of the vector fields becomes unbounded from below unless a Skyrme term is present with a coefficient satisfying

$$g_3 \geq g_1^2. \quad (3.6)$$

Analogous inequalities can also be derived that relate other four-point couplings to three-point couplings involving axial mesons [27].

Although the full construction of the Hamiltonian is somewhat tedious, a quick way to obtain these inequalities can be obtained by looking at the terms in the Lagrangian that involve time-derivatives of the fields. For example, to obtain (3.6) we need the kinetic energies and fourth and sixth terms of (3.4). By completing the square we can write these in the form

$$\mathcal{L} = \frac{f_\pi^2}{4}\langle u_\mu u^\mu \rangle - \frac{1}{2}\left\langle \left(V_{\mu\nu} + \frac{i}{2}g_1[u_\mu, u_\nu] \right)^2 \right\rangle + \frac{1}{8}(g_3 - g_1^2)\langle [u_\mu, u_\nu]^2 \rangle + \dots \quad (3.7)$$

The Skyrme term contains a positive-definite piece of second order in both the time and space derivatives of the pion fields. If the spatial gradients of these are large enough, this term will dominate over the normal kinetic term which is also quadratic in the time derivatives. For the overall coefficient of these is to remain positive for all field configurations, the coefficient of the Skyrme term must not be negative and hence the couplings must satisfy (3.6).

This unwanted behaviour can also show up in scattering processes as violations of unitarity. In an effective theory of π and ρ mesons only, such as (3.4), high-energy $\pi\pi$ scattering can violate the Froissart bound [44] (a consequence of unitarity) unless the coefficient of the Skyrme term satisfies the equality in (3.6) [11].⁴ This value for g_3 is the one that follows from resonance saturation by the ρ meson [24,10,11,25]. The equalities are thus consequences of the strong assumption of resonance saturation, namely that only one meson contributes in each of the relevant channels for $\pi\pi$ and other scattering processes. More generally, if several states with the same quantum numbers contribute, only the inequalities are satisfied.

Although it is rather surprising that a low-energy effective theory of this type should still give reasonable behaviour for high-energy processes, the values of the couplings that follow from this assumption do agree with those determined from low-energy scattering processes. The reasons for its success, like those of vector dominance and KSFR, remain mysterious.

Electromagnetic couplings of the pions and spin-1 mesons can be introduced in this scheme as described in Sec. IID. The usual minimal couplings of the photon to the vector current arise from the use of the field gradients of (2.38). Nonminimal couplings can include, for example, an anomalous magnetic moment for the ρ , described by a term of the form $\langle [V_\mu, V_\nu] F_+^{\mu\nu} \rangle$.

There can also be $\gamma\rho$ mixing, which is of interest in the context of vector dominance. At

⁴For a more detailed discussion of dispersion relations for $\pi\pi$ scattering and their relation to the parameters of ChPT, see [45].

lowest-order, this is described by the term

$$\mathcal{L}_{\rho\gamma} = -f_1 \langle V_{\mu\nu} F_+^{\mu\nu} \rangle, \quad (3.8)$$

which is chirally and gauge invariant. As well as providing $\gamma\rho$ mixing, this term contributes to the decay $\rho^0 \rightarrow \pi^+\pi^-\gamma$, where its effects can be seen near the endpoint in the photon spectrum [46].

Such a kinetic mixing is of order p^2 and so vanishes in the light- ρ limit used to obtain the KSFR relation in Sec. IIA. As we have already seen, the leading $\rho\pi\pi$ coupling in this scheme is of order p^3 , hence $g_{\gamma\rho}$ and $g_{\rho\pi\pi}$ both vanish as the ρ mass is taken to zero, while all the coupling constants are held fixed. This confirms that the KSFR relation (2.16) is indeed satisfied in the WCCWZ approach, although not in a way that has any practical use, because of the strong momentum dependence of the couplings.

With mixing of the form (3.8), the effective theory can display unphysical short-distance behaviour, analogous to that discussed above. In this case the conflict is not with general principles, such as unitarity or existence of a vacuum, but with QCD predictions for the behaviour of current-current correlators at high momentum [47]. For example, the Lagrangian (3.4) with mixing (3.8) (as well as minimal couplings) gives a pion electromagnetic form factor of

$$F_\pi(q^2) = 1 + \frac{g_1 f_1}{f_\pi^2} \frac{q^4}{M_V^2 - q^2}. \quad (3.9)$$

This grows at large q^2 as does the corresponding contribution of $\pi\pi$ intermediate states to the current-current correlator. This is inconsistent with the QCD expectation that it tend to a constant.

The cure is again to include extra terms in the Lagrangian, in this case [11]

$$\mathcal{L}_{\gamma\pi(nm)} = -\frac{i}{2} f_2 \langle F_+^{\mu\nu} [u_\mu, u_\nu] \rangle - \frac{1}{2} f_3 \langle F_{+\mu\nu} F_+^{\mu\nu} \rangle. \quad (3.10)$$

The second term here is needed to correct the unphysical contribution from the ρ meson to the current-current correlator. As above, an easy way to determine the coefficients for these

terms is to complete the square, and write the relevant pieces of the Lagrangian as

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2} \left\langle \left(V_{\mu\nu} + \frac{i}{2} g_1 [u_\mu, u_\nu] + f_1 F_{+\mu\nu} \right)^2 \right\rangle - \frac{1}{2} (f_3 - f_1^2) \langle F_+^{\mu\nu} F_{+\mu\nu} \rangle \\ & - \frac{i}{2} (f_2 - g_1 f_1) \langle [u_\mu, u_\nu] F_+^{\mu\nu} \rangle + \dots.\end{aligned}\quad (3.11)$$

In this form the momentum-dependent $\gamma\rho$ mixing could be removed by a change of variables [2,3,38]. However we do not have to perform this transformation here; all we need to note is that the unwanted high-momentum behaviour now arises only from the final two terms. It can thus be eliminated if their coefficients are zero. For example, the corresponding expression for the pion form factor is

$$F_\pi(q^2) = 1 + \frac{g_1 f_1}{f_\pi^2} \frac{q^4}{M_V^2 - q^2} + \frac{f_2}{f_\pi^2} q^2, \quad (3.12)$$

from which we can see that f_2 should satisfy

$$f_2 = g_1 f_1, \quad (3.13)$$

if the form factor is not to grow at large q^2 . Similar arguments can be used to show that f_3 should be given by

$$f_3 = f_1^2. \quad (3.14)$$

Implicit in these relations is the assumption of resonance saturation, since f_2 and f_3 should in principle also contain contributions from other states with the quantum numbers of the ρ . Under similar assumptions about the axial couplings, involving the a_1 meson, there can be another contribution to f_3 , of opposite sign [11].

The stronger assumption of complete vector dominance in the form factor [2,3] would require that $F_\pi(q^2)$ be given by

$$F_\pi(q^2) = \frac{M_V^2}{M_V^2 - q^2}. \quad (3.15)$$

This expression (3.12) can be put into this form if the coupling constants and ρ mass are related by

$$g_1 f_1 = \frac{f_\pi^2}{M_V^2}, \quad (3.16)$$

in addition to (3.13). This is the WCCWZ analogue of the relation (2.17) in models with minimal momentum dependence in the ρ couplings.

The full WCCWZ Lagrangian also contains an anomalous part, whose form I shall not discuss in detail. This includes terms like the photon coupling to the topological baryon current (2.37), which can be expressed in the form

$$B^\mu = \frac{i}{24\pi^2} \epsilon^{\mu\nu\kappa\lambda} \langle u_\nu u_\kappa u_\lambda \rangle. \quad (3.17)$$

There can also be a coupling of the isoscalar ω meson to this current, as well as further anomalous terms involving both ρ and ω mesons. Within the WCCWZ framework, there is no danger that such terms can violate the low-energy theorems for processes such as $\pi^0 \rightarrow \gamma\gamma$ or $\gamma \rightarrow 3\pi$ because of the $\mathcal{O}(p^2)$ nature of the photon-vector-meson mixing.

IV. HIDDEN-GAUGE THEORIES

In the simplest version of the hidden-gauge approach, the gauge symmetry is just $SU(2)$ and only vector mesons are treated as gauge bosons [35]. This symmetry is introduced by factorising $U(x)$ into two $SU(2)$ matrices [29,6],

$$U(x) = \xi_L(x)^\dagger \xi_R(x). \quad (4.1)$$

Since at each point in space-time this factorisation is arbitrary, these fields are invariant under local $SU(2)$ rotations. A gauge symmetry has thus been created by this choice of variables.

The extension to axial-vector mesons requires a local $SU(2) \times SU(2)$ symmetry, which can be introduced by writing $U(x)$ as a product of three unitary matrices [29,6,7],

$$U(x) = \xi_L(x)^\dagger \xi_M(x) \xi_R(x). \quad (4.2)$$

In this case, the new variables are symmetric under

$$\xi_R(x) \rightarrow h_R(x) \xi_R(x) g_R^\dagger$$

$$\begin{aligned}\xi_L(x) &\rightarrow h_L(x)\xi_L(x)g_L^\dagger \\ \xi_M(x) &\rightarrow h_L(x)\xi_M(x)h_R^\dagger(x),\end{aligned}\tag{4.3}$$

where $h_{L,R}(x)$ are SU(2) matrices with arbitrary x -dependence. The freedom to make space-time dependent rotations of $\xi_{r,l,m}(x)$ in this way provides the local SU(2) \times SU(2) symmetry of this scheme. This can be reduced to the simpler case of a local SU(2) symmetry by setting $\xi_M(x) = 1$.

One can always choose to work in the unitary gauge where

$$\xi_R(x) = \xi_L^\dagger(x) = u(x), \quad \xi_M(x) = 1,\tag{4.4}$$

for all x . The symmetry (4.3) then reduces to the usual nonlinear realisation of chiral symmetry [8,9], as in (2.23), where the x dependence of $h(x) = h_R(x) = h_L(x)$ is no longer arbitrary but is given in terms of the pion fields. This gauge fixing thus provides the basis for translating between the hidden-gauge and WCCWZ formalisms.

In this approach, spin-1 fields are introduced as gauge bosons of this artificial local symmetry. Right- and left-handed gauge fields transform under the symmetry as, respectively,

$$\begin{aligned}\widehat{X}_\mu(x) &\rightarrow h_R(x)\widehat{X}_\mu(x)h_R^\dagger(x) + \frac{i}{\sqrt{2}g}h_R(x)\partial_\mu h_R^\dagger(x) \\ \widehat{Y}_\mu(x) &\rightarrow h_L(x)\widehat{Y}_\mu(x)h_L^\dagger(x) + \frac{i}{\sqrt{2}g}h_L(x)\partial_\mu h_L^\dagger(x),\end{aligned}\tag{4.5}$$

where I use hats to distinguish the hidden-gauge spin-1 fields from those of the WCCWZ approach. The corresponding gauge-covariant field strengths are

$$\begin{aligned}\widehat{X}_{\mu\nu} &= \partial_\mu\widehat{X}_\nu - \partial_\nu\widehat{X}_\mu - i\sqrt{2}g[\widehat{X}_\mu, \widehat{X}_\nu] \\ \widehat{Y}_{\mu\nu} &= \partial_\mu\widehat{Y}_\nu - \partial_\nu\widehat{Y}_\mu - i\sqrt{2}g[\widehat{Y}_\mu, \widehat{Y}_\nu].\end{aligned}\tag{4.6}$$

It is usually more convenient to work in terms of the vector and axial fields, $\widehat{V}_\mu = (\widehat{X}_\mu + \widehat{Y}_\mu)/\sqrt{2}$ and $\widehat{A}_\mu = (\widehat{X}_\mu - \widehat{Y}_\mu)/\sqrt{2}$. The field strengths for these are

$$\widehat{V}_{\mu\nu} = \partial_\mu\widehat{V}_\nu - \partial_\nu\widehat{V}_\mu - ig[\widehat{V}_\mu, \widehat{V}_\nu] - ig[\widehat{A}_\mu, \widehat{A}_\nu]$$

$$\widehat{A}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - ig[\widehat{V}_\mu, \widehat{A}_\nu] - ig[\widehat{A}_\mu, \widehat{V}_\nu]. \quad (4.7)$$

The gauge-covariant first derivatives of the pion fields are

$$\begin{aligned} R_\mu &= -i \left[(\partial_\mu \xi_L) \xi_L^\dagger - i\sqrt{2}g \widehat{X}_\mu \right] \\ L_\mu &= -i \left[(\partial_\mu \xi_R) \xi_R^\dagger - i\sqrt{2}g \widehat{Y}_\mu \right] \\ M_\mu &= -i \left[(\partial_\mu \xi_M) \xi_M^\dagger + i\sqrt{2}g \xi_M \widehat{X}_\mu \xi_M^\dagger - i\sqrt{2}g \widehat{Y}_\mu \right]. \end{aligned} \quad (4.8)$$

Of these, R_μ transforms covariantly under the right-handed local symmetry, L_μ and M_μ under the left-handed. A general gauge-invariant Lagrangian in this approach consists of all terms that can be constructed out of traces of products of R_μ , L_μ , M_μ , $\widehat{X}_{\mu\nu}$, $\widehat{Y}_{\mu\nu}$, and their covariant derivatives, and that are symmetric under parity. Factors of ξ_M and ξ_M^\dagger should be inserted between right- and left-covariant quantities. Writing out explicitly only terms of second order in the pion field gradients (which also provide mass terms for the heavy mesons) and the vector-meson kinetic terms, one has the Lagrangian [29,6,7]

$$\begin{aligned} \mathcal{L} &= \frac{af_\pi^2}{4} \langle (L_\mu + \xi_M R_\mu \xi_M^\dagger)^2 \rangle + \frac{bf_\pi^2}{4} \langle (L_\mu - \xi_M R_\mu \xi_M^\dagger)^2 \rangle \\ &+ \frac{cf_\pi^2}{4} \langle M_\mu M^\mu \rangle + \frac{df_\pi^2}{4} \langle (L_\mu - \xi_M R_\mu \xi_M^\dagger - M_\mu)^2 \rangle \\ &- \frac{1}{2} \langle \widehat{X}_{\mu\nu} \widehat{X}^{\mu\nu} + \widehat{Y}_{\mu\nu} \widehat{Y}^{\mu\nu} \rangle + \dots \end{aligned} \quad (4.9)$$

In the unitary gauge defined by (4.4), the transformation properties of the spin-1 fields are

$$\begin{aligned} \widehat{V}_\mu &\rightarrow h \widehat{V}_\mu h^\dagger + \frac{i}{g} h \partial_\mu h^\dagger \\ \widehat{A}_\mu &\rightarrow h \widehat{A}_\mu h^\dagger, \end{aligned} \quad (4.10)$$

where h is the compensating $SU(2)$ rotation of the nonlinear realisation of chiral symmetry, (2.23). The covariant gradients of (4.8) reduce to

$$\begin{aligned} R_\mu &= iu^\dagger \partial_\mu u - g(\widehat{V}_\mu + \widehat{A}_\mu) \\ L_\mu &= iu \partial_\mu u^\dagger - g(\widehat{V}_\mu - \widehat{A}_\mu) \end{aligned}$$

$$M_\mu = 2g\hat{A}_\mu. \quad (4.11)$$

Since these can be combined to give

$$\begin{aligned} R_\mu + L_\mu &= 2(i\Gamma_\mu - g\hat{V}_\mu) \\ R_\mu - L_\mu &= u_\mu - 2g\hat{A}_\mu, \end{aligned} \quad (4.12)$$

we find that \hat{V}_μ always appears in the combination

$$V_\mu = \hat{V}_\mu - \frac{i}{g}\Gamma_\mu. \quad (4.13)$$

This transforms homogeneously under the nonlinear chiral rotation, as can be seen from (2.27) and (4.10).

In the unitary gauge we can therefore change variables to the vector field V_μ of (4.13) to obtain a Lagrangian of the WCCWZ type. (The axial field already transforms homogeneously in this gauge, (4.10).) With the aid of (2.31), the field strengths (4.7) can be expressed in terms of the new fields as

$$\begin{aligned} \hat{V}_{\mu\nu} &= V_{\mu\nu} + \frac{i}{4g}[u_\mu, u_\nu] - ig[V_\mu, V_\nu] - ig[A_\mu, A_\nu] \\ \hat{A}_{\mu\nu} &= A_{\mu\nu} - ig[V_\mu, A_\nu] - ig[A_\mu, V_\nu], \end{aligned} \quad (4.14)$$

where the covariant field gradients are defined in (3.1) above. Terms involving higher gauge-covariant derivatives can be rewritten in terms of the covariant derivative (2.29) using

$$\widehat{D}_\mu = \partial_\mu - ig[\hat{V}_\mu, \] = \nabla_\mu - ig[V_\mu, \]. \quad (4.15)$$

Each term of the general hidden-gauge Lagrangian in the unitary gauge has a corresponding term in the general WCCWZ Lagrangian, where $\hat{V}_\mu - i\Gamma_\mu/g$ has been replaced by V_μ , \widehat{D}_μ by ∇_μ , $\hat{V}_{\mu\nu}$ by $V_{\mu\nu}$, and $\hat{A}_{\mu\nu}$ by $A_{\mu\nu}$. The coupling constants will not be identical but, if one takes account of Eqs. (4.14,15), there is a well defined way to convert from one approach to the other. This generalises Georgi's observation [28] of the equivalence of the two formalisms to the case of axial as well as vector fields.

An important feature to note is that the gauge coupling constant g of the hidden-gauge approach does not appear in the WCCWZ approach. Indeed hidden-gauge Lagrangians with different values of g , and containing different higher-order interactions, can be equivalent to the same WCCWZ theory. This should not be too surprising: the local symmetry is not physical but arises from a particular choice of field variables in (4.1,2) and hence the corresponding coupling is not a physical quantity. The significance of g becomes clearer if one starts from a WCCWZ Lagrangian and converts it into a hidden-gauge one using (4.13) in reverse. Any value of g can be used in (4.13) to define a new vector field \widehat{V}_μ and the resulting Lagrangian will have the form of a hidden-gauge theory in the unitary gauge. Different choices of g therefore correspond to different choices of vector field. The value of g should thus be fixed by considerations of calculational convenience, for example the elimination of certain types of term from the effective Lagrangian.

To explore this equivalence in more detail, let us examine a specific hidden-gauge theory. The example considered is the most commonly used hidden-gauge model, introduced by Bando *et al.* [35]. This contains a vector but no axial field and so is invariant under the diagonal SU(2) subgroup of the local symmetry only. Its Lagrangian has the form

$$\mathcal{L} = \frac{f_\pi^2}{4} \langle (L_\mu - R_\mu)^2 \rangle + \frac{af_\pi^2}{4} \langle (L_\mu + R_\mu)^2 \rangle - \frac{1}{2} \langle \widehat{V}_{\mu\nu} \widehat{V}^{\mu\nu} \rangle, \quad (4.16)$$

where

$$\begin{aligned} R_\mu &= -i \left[(\partial_\mu \xi_L) \xi_L^\dagger - ig \widehat{V}_\mu \right] \\ L_\mu &= -i \left[(\partial_\mu \xi_R) \xi_R^\dagger - ig \widehat{V}_\mu \right]. \end{aligned} \quad (4.17)$$

In the unitary gauge this becomes

$$\mathcal{L} = \frac{f_\pi^2}{4} \langle u_\mu u^\mu \rangle + af_\pi^2 \langle (i\Gamma_\mu - g\widehat{V}_\mu)^2 \rangle - \frac{1}{2} \langle \widehat{V}_{\mu\nu} \widehat{V}^{\mu\nu} \rangle, \quad (4.18)$$

showing that the ρ mass is given in terms of the parameter a by

$$m_V^2 = ag^2 f_\pi^2. \quad (4.19)$$

The replacement of the covariant derivatives (3.1) by gauge-covariant ones (4.7), means that the Weinberg-Tomozawa piece of $\pi\rho$ scattering no longer appears in the kinetic term

for the ρ in the Lagrangian. Instead it is generated entirely from ρ exchange. The fact that this leads to the current-current interaction (2.13) can be seen from the $\rho\pi\pi$ coupling contained in (4.18),

$$-2igaf_\pi^2\langle\widehat{V}^\mu\Gamma_\mu\rangle=\frac{1}{2}ag\widehat{\mathbf{V}}^\mu\cdot\boldsymbol{\pi}\wedge\partial_\mu\boldsymbol{\pi}+\mathcal{O}(\pi^4), \quad (4.20)$$

and the 3ρ coupling

$$2ig\langle(\partial_\mu\widehat{V}_\nu-\partial_\nu\widehat{V}_\mu)[\widehat{V}^\mu,\widehat{V}^\nu]\rangle=-g\widehat{\mathbf{V}}^\mu\cdot\widehat{\mathbf{V}}^\nu\wedge\partial_\mu\widehat{\mathbf{V}}_\nu. \quad (4.21)$$

Moreover the factor a in the $\rho\pi\pi$ coupling cancels with that from the ρ mass (4.19) in the denominator. The low-energy theorem is thus satisfied independently of the value of the ρ mass.

Using (4.13,14) the Lagrangian (4.18) can be expressed in the equivalent WCCWZ form (3.4), with the following values for the coupling constants:

$$g_1=\frac{1}{2g}, \quad g_2=2g, \quad g_3=\frac{1}{4g^2}, \quad g_4=1, \quad g_5=4g^2. \quad (4.22)$$

The couplings in this model thus satisfy the relations

$$g_3=g_1^2, \quad g_4=g_1g_2, \quad g_5=g_2^2, \quad (4.23)$$

which arise from assuming resonance saturation by the ρ -meson, as discussed in Sec. III. The other condition that defines the model is a relation between the $\rho\pi\pi$ and 3ρ couplings,

$$g_1=\frac{1}{g_2}. \quad (4.24)$$

These relations (4.23,24) allow the three- and four-point couplings to be combined into a kinetic term for the vector field with a Yang-Mills form. Note that all of these relations hold for any value of the ρ mass (or equivalently of the parameter a).

The replacement of derivatives by gauge-covariant ones involving the external fields generates all the usual minimal electromagnetic couplings. The presence in the connection (2.38) of a term linear in the vector field v_μ means that the mass term in (4.18) develops

a $\gamma\rho$ mixing piece. The presence of such a momentum-independent mixing term makes the hidden-gauge formalism an especially convenient one for embodying the idea of vector dominance. The coefficient of this mixing term is agf_π^2 . By comparing this with the $\rho\pi\pi$ coupling (4.20), we see that the KSFR relation (2.16) is satisfied [29,30], as it should be since the Lagrangian has been constructed to respect chiral symmetry. It is unsurprising that this low-energy theorem continues to hold when loop corrections are included [48,49].

Other couplings can be obtained from the corresponding WCCWZ forms by making the replacements (4.13,14) above. For example, an additional, momentum-dependent $\gamma\rho$ mixing can be described by a term of the form $\langle \hat{V}_{\mu\nu} F_+^{\mu\nu} \rangle$.

As an example, consider an extension of the model (4.16) with only the minimal replacement of derivatives. In converting this into its WCCWZ equivalent, we need to note that the presence of the field tensor in (2.40) produces an additional term in the relation corresponding to (4.14),

$$\hat{V}_{\mu\nu} = V_{\mu\nu} + \frac{i}{4g}[u_\mu, u_\nu] + \frac{i}{g}F_{+\mu\nu} - ig[V_\mu, V_\nu]. \quad (4.25)$$

This leads to a $\gamma\rho$ mixing of the form (3.7) together with the terms given in (3.10) and an anomalous magnetic moment for the ρ . The coupling constants for these are all related to g ,

$$f_1 = \frac{1}{g}, \quad f_2 = \frac{1}{2g^2}, \quad f_3 = \frac{1}{g^2}. \quad (4.26)$$

The model thus satisfies the resonance saturation conditions (3.13,14) for the electromagnetic couplings, in addition to those of (4.23).

There is one other relation between the coupling constants that defines this model. This is a connection between the $\rho\pi\pi$ and $\gamma\rho$ couplings [11],

$$f_1 = 2g_1, \quad (4.27)$$

which does not follow from resonance saturation. The factor of two in this relation makes it very reminiscent of the KSFR relation (2.16). Although it refers to the momentum-dependent couplings of the WCCWZ formalism and not those appearing in (2.16), this relation is nonetheless very closely related to the KSFR one, as we shall see below.

Vector dominance of the pion electromagnetic form factor is obtained if the photon coupling to the pion current in the first term of (4.18) cancels with the corresponding piece of the second term. The photon then couples to the pion only through a virtual ρ . This will happen if the ρ mass satisfies

$$m_V^2 = 2g^2 f_\pi^2, \quad (4.28)$$

which is just the KSFR relation in its second, vector-dominance form (2.18). In terms of the parameters of Bando *et al.* [35,7] this corresponds to $a = 2$. Vector dominance in the couplings of the photon to all hadrons requires universal coupling of the ρ meson to the conserved isospin current. From the expressions for the $\rho\pi\pi$ and 3ρ couplings (4.20,21), we can see that for $a = 2$ these have the same strength and so this model also embodies universal coupling of the ρ to itself and to the pion.

In looking at predictions of the model (4.16), consequences of the hidden-gauge choice of field should not be confused with those arising from the relations between the coupling constants (4.23,24,26,27). The former controls merely the form of off-shell extrapolations of those amplitudes. The latter lead to relations between amplitudes for physical processes, and are of course specific to the choice of Lagrangian. For example the relation (4.24) between the $\rho\pi\pi$ and 3ρ couplings can be removed without violating the hidden-gauge invariance by adding a term of the form $\langle \hat{V}_{\mu\nu} [R^\mu + L^\mu, R^\nu + L^\nu] \rangle$ to the Lagrangian (4.16). This is invariant under the same local SU(2) symmetry as the rest of the Lagrangian. It provides an additional contribution to the 3ρ coupling beyond that in the kinetic term. After gauge-fixing and change of variables, such a term would lead to an equivalent WCCWZ Lagrangian that would not satisfy (4.24).

To see why the specific choice of field in (4.18) is particularly convenient, remember that the $\rho\pi\pi$ of the corresponding WCCWZ Lagrangian (3.5) is of order p^3 . Also the $\gamma\rho$ mixing (3.8) is of order p^2 . In contrast, when the model is expressed in hidden-gauge form, the leading $\rho\pi\pi$ coupling (4.20) is of first order in the momenta of the particles involved and the $\gamma\rho$ mixing is independent of momentum. Using the field defined by (4.13) with the constant

g given by

$$g = \frac{1}{2g_1} \quad (4.29)$$

eliminates any $\mathcal{O}(p^3)$ $\rho\pi\pi$ term from the hidden-gauge Lagrangian. This happens even if the initial WCCWZ Lagrangian contains other, higher-derivative $\rho\pi\pi$ couplings. The advantage of the hidden-gauge choice of field with this g is that any corrections to the leading $\rho\pi\pi$ of (4.20) are of order p^5 or higher. Provided m_V is small compared with the scale at which physics beyond the $\pi\rho$ Lagrangian becomes significant, the momentum dependence of the effective $\rho\pi\pi$ coupling should be small in the hidden-gauge representation.

Moreover, if the $\gamma\rho$ mixing strength of the WCCWZ Lagrangian satisfies the condition (4.27) then the same choice of field also eliminates any $\mathcal{O}(p^2)$ mixing term from the hidden-gauge Lagrangian. Corrections to it are thus at least of order p^4 . Hence the leading corrections to both of the coupling constants that appear in (2.16) are of order p^4 instead of p^2 in this model. The model (4.16) thus embodies the empirical observation that the KSFR relation in its first version (2.16), which relates the $\rho\pi\pi$ coupling and $\gamma\rho$ mixing at zero four-momentum, is actually well satisfied by the values for on-shell ρ mesons.⁵

The relation (4.27) between the couplings of the equivalent WCCWZ theory is what makes it possible for the KSFR relation to be satisfied on-shell as well as at zero four-momentum. However one should remember that the manipulations here shed no light on the origin of this. Like vector dominance, it remains an phenomenologically successful assumption that arises neither from chiral symmetry nor from the assumption of resonance saturation.

⁵I am assuming here that the order p^4 corrections are small compared to the leading terms in the expansion. In principle the KSFR relation could also be satisfied if the higher-order contributions are large but remain in the same 2:1 ratio as those in (2.16) and (4.27).

V. MASSIVE YANG-MILLS

The massive Yang-Mills approach [4–6] is based on vector and axial fields that transform linearly under the $SU(2) \times SU(2)$ symmetry, as in (2.22). The Lagrangian for these is chosen to contain kinetic terms of the Yang-Mills form, including three- and four-point interactions. The couplings of the spin-1 fields to pions are also chosen to have a gauge-invariant form, ensuring universal coupling of the ρ and allowing photons to be coupled in a way consistent with vector dominance. Although the interaction terms respect a local $SU(2) \times SU(2)$ symmetry, the full theory does not since it also includes mass terms for the spin-1 mesons.

A simple massive Yang-Mills Lagrangian, which illustrates the features of the approach, consists of the gauged sigma model (a nonlinear version of the model used by Gasiorowicz and Geffen [4])

$$\begin{aligned} \mathcal{L} = & \frac{f_0^2}{4} \langle \widetilde{D}_\mu U (\widetilde{D}_\mu U)^\dagger \rangle - \frac{1}{2} \langle \widetilde{X}_{\mu\nu} \widetilde{X}^{\mu\nu} + \widetilde{Y}_{\mu\nu} \widetilde{Y}^{\mu\nu} \rangle \\ & + m_V^2 \langle \widetilde{X}_\mu \widetilde{X}^\mu + \widetilde{Y}_\mu \widetilde{Y}^\mu \rangle, \end{aligned} \quad (5.1)$$

where

$$\widetilde{D}_\mu U = \partial_\mu U + i\sqrt{2}gU\widetilde{X}_\mu - i\sqrt{2}g\widetilde{Y}_\mu U, \quad (5.2)$$

and the field strengths $\widetilde{X}_{\mu\nu}$, $\widetilde{Y}_{\mu\nu}$ are defined analogously to those in (4.6).

In this approach, low-energy theorems cannot simply be read off from terms in the Lagrangian; they are obtained from combinations of a number of pieces. A further complication is the presence of a πa_1 mixing term which can be removed by an appropriate shift in the definition of the axial field [4,11]. The gauge-like couplings of the ρ mean that (t -channel) ρ exchange contributes to the Weinberg-Tomozawa term for $\pi\rho$ scattering. However one has also to include pieces with intermediate (s - and u -channel) π and a_1 states to satisfy the low-energy theorem. The isospin-symmetric pion scattering amplitude, which should vanish for at threshold in the chiral limit (*cf.* [8]), is also rather complicated. It contains a momentum-independent contribution from the pion kinetic term of (5.1), but this is exactly cancelled by contributions involving intermediate π and a_1 states [50]. All of this shows that, in the

massive Yang-Mills formalism, one has to take care to include all possible contributions to any process as otherwise low-energy theorems will be violated.

The massive Yang-Mills theory can be converted into an equivalent WCCWZ one by using $u(x)$ to construct spin-1 fields that transform under the nonlinear realisation of chiral symmetry as described in Sec. IIC. The new fields are given by (2.33) and the kinetic terms can be expressed in terms of their covariant derivatives, defined as in (3.1), using

$$\begin{aligned}\widetilde{X}_{\mu\nu} &= u^\dagger \left[X_{\mu\nu} + \frac{i}{2}[u_\mu, X_\nu] - \frac{i}{2}[u_\nu, X_\mu] - i\sqrt{2}\widetilde{g}[X_\mu, X_\nu] \right] u \\ \widetilde{Y}_{\mu\nu} &= u^\dagger \left[Y_{\mu\nu} - \frac{i}{2}[u_\mu, Y_\nu] + \frac{i}{2}[u_\nu, Y_\mu] - i\sqrt{2}\widetilde{g}[Y_\mu, Y_\nu] \right] u.\end{aligned}\tag{5.3}$$

In terms of u_μ and these fields, the pion kinetic term can be written

$$\langle \widetilde{D}_\mu U (\widetilde{D}_\mu U)^\dagger \rangle = \langle [u_\mu - \sqrt{2}\widetilde{g}(X_\mu - Y_\mu)]^2 \rangle.\tag{5.4}$$

This contains the πa_1 mixing term mentioned above. To remove this, it is convenient to define WCCWZ vector and axial fields by

$$\begin{aligned}V_\mu &= \frac{1}{\sqrt{2}}(X_\mu + Y_\mu) \\ A_\mu &= \frac{1}{\sqrt{2}}(X_\mu - Y_\mu) - \frac{\widetilde{g}f_0^2}{2m_A^2}u_\mu.\end{aligned}\tag{5.5}$$

The kinetic terms for the spin-1 fields can then be expressed in terms of V_μ and A_μ and their covariant derivatives (3.1), making use of (2.30). The Lagrangian (5.1) then takes the form

$$\begin{aligned}\mathcal{L} &= \frac{f_\pi^2}{4} \langle u_\mu u^\mu \rangle + m_V^2 \langle V_\mu V^\mu \rangle + m_A^2 \langle A_\mu A^\mu \rangle \\ &\quad - \frac{1}{2} \left\langle \left\{ V_{\mu\nu} - i\widetilde{g}[V_\mu, V_\nu] - i\widetilde{g}[A_\mu, A_\nu] \right. \right. \\ &\quad \quad \left. \left. + i\frac{1}{2}Z^2([u_\mu, A_\nu] - [u_\nu, A_\mu]) \right. \right. \\ &\quad \quad \left. \left. + \frac{i}{4\widetilde{g}}(1 - Z^4)[u_\mu, u_\nu] \right\}^2 \right\rangle \\ &\quad - \frac{1}{2} \left\langle \left\{ A_{\mu\nu} - i\widetilde{g}[V_\mu, A_\nu] - i\widetilde{g}[A_\mu, V_\nu] \right. \right.\end{aligned}$$

$$+i\frac{1}{2}Z^2\left([u_\mu, V_\nu] - [u_\nu, V_\mu]\right)\bigg\}^2\bigg\rangle. \quad (5.6)$$

where

$$Z^2 = 1 - \frac{\tilde{g}^2 f_0^2}{m_A^2} = 1 - \frac{\tilde{g}^2 f_\pi^2}{m_V^2}, \quad (5.7)$$

and the physical pion decay constant is given by [6]

$$f_\pi^2 = f_0^2 Z^2, \quad (5.8)$$

and the a_1 mass by

$$m_A^2 = m_V^2 / Z^2. \quad (5.9)$$

Although I have demonstrated the equivalence here for only the theory defined by (5.1), it is general. Any massive Yang-Mills Lagrangian can be expressed in WCCWZ form using (5.2–5). Conversely, any WCCWZ Lagrangian can be converted into an equivalent massive Yang-Mills theory by inverting these changes of variable. Of course the resulting Lagrangian can contain many terms beyond those present in (5.1), including many non-gauge-invariant interactions. Combined with the results of Section III, this reproduces and generalises the well-known equivalence of the hidden-gauge and massive Yang-Mills formalisms [51–53,6].

By comparing the terms in the Lagrangian (5.6) with the corresponding ones in (3.4), we can see that the couplings satisfy the relations (4.23) arising from assuming resonance saturation of the four-point interactions. This is similar to the hidden-gauge theory defined by (4.16). The two theories are thus closely related although obviously not identical: the massive Yang-Mills one contains an axial as well as a vector field, and its $\rho\pi\pi$ and 3ρ couplings do not satisfy (4.24). The latter is a consequence of the the additional momentum-dependent $\rho\pi\pi$ couplings that appear after diagonalising in the πa_1 sector. One can always cancel out this momentum dependence by adding extra nonminimal terms to the massive Yang-Mills Lagrangian [5].⁶ The resulting massive Yang-Mills Lagrangian is then exactly

⁶One can also force the theory into exact equivalence with the one of Bando et al. [35] by imposing

equivalent to the hidden-gauge one involving axial as well as vector mesons introduced in Ref. [30].

Electromagnetic couplings can be included in the usual way by replacing derivatives with gauge-covariant derivatives involving the external fields. Nonminimal terms can be constructed, with a little care to ensure that they respect chiral invariance. For example, $\gamma\rho$ mixing can be obtained from a combination of two kinetic mixing terms, $\langle X_{\mu\nu}(v^{\mu\nu} + a^{\mu\nu}) + Y_{\mu\nu}(v^{\mu\nu} - a^{\mu\nu}) \rangle$ and $\langle X_{\mu\nu}U^\dagger(v^{\mu\nu} - a^{\mu\nu})U + Y_{\mu\nu}U(v^{\mu\nu} + a^{\mu\nu})U^\dagger \rangle$.

Vector dominance can be realised if the ρ mass satisfies the second form of the KSFR relation (2.18). In this case $Z = 1/2$ and the ρ and a_1 masses satisfy the Weinberg relation $m_A = \sqrt{2}m_V$ [55]. Also, with this value of the ρ mass, ρ exchange generates the whole of the Weinberg-Tomozawa term.

For completeness, I should mention the approach suggested by Brihaye, Pak and Rossi [31] and investigated further by Kuraev, Silagadze and coworkers [56,15]. This is based on a Yang-Mills-type coupling of the ρ , as in the Lagrangian (5.1), but without a chiral partner a_1 -field. Simply omitting the axial field from that Lagrangian leaves a theory that is not chirally symmetric. However as described in Ref. [31] additional counterterms can be added to that Lagrangian to ensure that low-energy theorems arising from chiral symmetry are maintained.

Such a theory can be generated by taking a hidden-gauge Lagrangian, such as that of (4.18), and reversing the procedure above for converting fields that transform linearly under chiral symmetry into ones that transform nonlinearly. Specifically one can define a new vector field \tilde{V}_μ , related to the hidden-gauge field \hat{V}_μ by

$$\hat{V}_\mu = \frac{1}{2} \left(u^\dagger \tilde{V}_\mu u + u \tilde{V}_\mu u^\dagger \right). \quad (5.10)$$

Note that this \tilde{V}_μ has no axial partner and so does not transform in any simple way under

a suitable constraint on the axial field [54,6]. This can be seen using the WCCWZ form (5.6): if one demands that $A_\mu = (Z^2/2g)u_\mu$ then one is left with the WCCWZ equivalent of (4.16).

chiral transformations. By adding and subtracting suitable terms, similar to those in (5.4), the pion kinetic term can be converted to a form involving gauge-covariant derivatives. The full Lagrangian is then

$$\begin{aligned} \mathcal{L} = & \frac{f_\pi^2}{4} \langle \widetilde{D}_\mu U (\widetilde{D}_\mu U)^\dagger \rangle + \frac{\widetilde{g} f_\pi^2}{2} \langle \widetilde{V}_\mu (u u_\mu u^\dagger - u^\dagger u_\mu u) \rangle - \frac{\widetilde{g}^2 f_\pi^2}{4} \langle (u^\dagger V_\mu u - u V_\mu u^\dagger)^2 \rangle \\ & + a f_\pi^2 \langle (i\Gamma_\mu - g\widehat{V}_\mu)^2 \rangle - \frac{1}{2} \langle \widehat{V}_{\mu\nu} \widehat{V}^{\mu\nu} \rangle. \end{aligned} \quad (5.11)$$

By choosing the coupling \widetilde{g} to be related to the parameters of the hidden-gauge Lagrangian by

$$\widetilde{g} = \frac{ag}{2}, \quad (5.12)$$

one can ensure that the $\mathcal{O}(p)$ $\rho\pi\pi$ coupling (4.20) in the fourth term of (5.11) is cancelled by that in the second term. The $\rho\pi\pi$ coupling is then given entirely by the gauge-covariant derivatives in the first term of (5.11).

The resulting theory has a Yang-Mills structure for the ρ kinetic energy and $\rho\pi\pi$ couplings, together with a ρ mass term and a number of additional couplings. These extra couplings are required if the low-energy theorems of chiral symmetry, such as (2.11), are to be satisfied. They include the counterterms discussed in Refs. [31,56] together with many others. For example the third term of (5.11) contains a momentum-independent $\rho\rho\pi\pi$ coupling. This term is omitted in the calculations of Kuraev *et al.* [56,15] but it is needed to cancel out a corresponding piece of the pion kinetic term, which would otherwise give a nonzero amplitude for $\pi\rho$ scattering at threshold in the chiral limit.

The consequences of failing to include all the necessary counterterms in this approach are illustrated by the calculations of the rare decays $\rho \rightarrow 2\pi^+\pi^-$ and $\rho \rightarrow 2\pi^0\pi^+\pi^-$, which have been proposed as possible tests of effective Lagrangians for vector mesons [14,15].⁷

⁷It should be clear that the formal equivalence between the various schemes described here means that there is no way to discriminate between them empirically. Nonetheless one might still hope to test assumptions about the values of some of the other couplings in particular Lagrangians, for example the 3ρ vertex [57].

The decay rates obtained from various Lagrangians that respect the constraints of chiral symmetry [57] are at least an order of magnitude smaller than those from Lagrangians that violate some of these constraints [14,15].

VI. TENSOR FIELDS

The other main formalism for inclusion of spin-1 mesons differs from those we have seen so far in that it uses antisymmetric tensor fields to describe these particles [10] (see [32] for the extension to the anomalous sector). The field describing the ρ and a_1 mesons are denoted here by $\bar{V}_{\mu\nu}$ and $\bar{A}_{\mu\nu}$, and these transform homogeneously under the nonlinear realisation of chiral symmetry in Sec. IIC. The mixed space-time components of these fields, \bar{V}_{0i} and \bar{A}_{0i} , describe the physical degrees of freedom. The others, like the time components of the vector fields, should be eliminated by constraints. The simplest Lagrangian of this form for pions and ρ mesons coupled to external fields is [10,11]

$$\mathcal{L} = \frac{f_\pi^2}{4} \langle u_\mu u^\mu \rangle - \frac{1}{2} \langle \nabla^\lambda \bar{V}_{\lambda\mu} \nabla_\nu \bar{V}^{\nu\lambda} \rangle + \frac{m_V^2}{2} \langle \bar{V}_{\mu\nu} \bar{V}^{\mu\nu} \rangle + \frac{i}{2} G_1 \langle \bar{V}_{\mu\nu} [u^\mu, u^\nu] \rangle + F_1 \langle \bar{V}_{\mu\nu} F_+^{\mu\nu} \rangle. \quad (6.1)$$

The covariant derivatives ∇_μ in the ρ kinetic term mean that the Weinberg-Tomozawa amplitude can be obtained directly from that term, as in the WCCWZ formalism. The coupling terms in (6.1) are analogous to terms in (3.4,8) but differ in that they involve one less power of momentum. Their contributions to decays of on-shell ρ mesons are identical if the coupling constants are related by [11]

$$G_1 = m_V g_1, \quad F_1 = m_V f_1. \quad (6.2)$$

Moreover the smaller momentum dependence of these interactions means that amplitudes calculated in this scheme satisfy the constraints arising from unitarity and QCD combined with resonance saturation. There is thus no need to supplement the Lagrangian (6.1) with four-point couplings of the type discussed in Sec. III. Hence, if resonance saturation is assumed, this scheme provides a particularly economical expression for the Lagrangian.

A complete proof of the equivalence of the WCCWZ and tensor-field schemes does not exist at present. For any term in the general WCCWZ Lagrangian one can write down an analogous one for the tensor fields. This is obvious for any interaction terms constructed out of the antisymmetric covariant derivatives (3.1), which can simply be replaced by the tensor fields. Other interactions can be generated by making the replacements

$$V_\mu \rightarrow -\frac{1}{m_V} \nabla^\lambda \overline{V}_{\lambda\mu}, \quad A_\mu \rightarrow -\frac{1}{m_V} \nabla^\lambda \overline{A}_{\lambda\mu}. \quad (6.3)$$

This replacement should also be used in any terms involving the symmetric covariant derivatives [33]. However beyond order p^4 any demonstration of the exact equivalence of the resulting Lagrangian to the original is complicated by the need to enforce constraints on the fields so that they describe only physical degrees of freedom. As result the equivalence of the two schemes has been shown only for terms up to order p^4 in the WCCWZ Lagrangian [11,33,34].

The substitution just described is purely a book-keeping device for enumerating the terms in the general tensor-field Lagrangian. Unlike the changes of variables described for the vector fields, it does not provide a means to convert a particular WCCWZ Lagrangian directly into tensor-field form or *vice versa*. In practice what is usually done is to integrate out the spin-1 fields from both approaches and then to compare the local terms in the resulting effective actions for pions only [11,33]. As an illustration of this, consider the Skyrme term. Integrating out the ρ in the WCCWZ case generates a contribution to the coefficient of this term of second-order in g_1 , in addition to the g_3 originally in the Lagrangian (5.4). The net coefficient should match with that in the tensor-field case, where it arises purely from any Skyrme term in the Lagrangian. If resonance saturation holds then $g_3 = g_1^2$ and the two contributions exactly cancel in the WCCWZ case. There is then no explicit Skyrme term in the equivalent tensor-field Lagrangian (6.1).

Recently, a path integral approach to re-expressing a vector-field theory in terms of tensor fields has been described in [34]. This is inspired by the dual transformation of gauge theories [58] and its starting point is a change of variables corresponding to (6.3) in the

path integral. This procedure does generate, for example, the change in the coefficient of the Skyrme term and in principle it could be used to translate a Lagrangian from WCCWZ to tensor-field formalism. However beyond the lowest-order terms of the Lagrangian (3.4,8) there are practical problems that arise from the constraints on the fields.

VII. SYMMETRY BREAKING

So far I have concentrated on the chirally symmetric parts of the Lagrangians in the various approaches. These can be straightforwardly extended to incorporate the effects of explicit symmetry breaking by the current quark masses. Such Lagrangians with vector mesons have been the subject of much recent discussion in the context of charge-symmetry breaking in the NN interaction. These effects arise because the difference between the up- and down-quark current masses breaks isospin as well as chiral symmetry. Much of this interest has focussed on the momentum-dependence of the mixing between the ρ and ω mesons (see [16,59] and references therein). The forms of possible $\rho\omega$ mixing vertices are analogous to those for $\gamma\rho$ mixing, with one exception: the absence of gauge invariance means that both mass and kinetic mixing terms are permitted in any formalism.

For example, such effects can be included in a WCCWZ Lagrangian with terms of the form

$$\begin{aligned} \mathcal{L} = & \frac{f_\pi^2}{4} \langle u_\mu u^\mu \rangle - \frac{1}{2} \langle V_{\mu\nu} V^{\mu\nu} \rangle + m_V^2 \langle V_\mu V^\mu \rangle - \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu \\ & - \frac{i}{2} g_1 \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle + \frac{1}{8} g_3 \langle [u_\mu, u_\nu]^2 \rangle + b_0 \langle \chi_+ \rangle - 2b_1 \langle \chi_+ V_\mu \rangle \omega^\mu \\ & - b_2 \langle \chi_+ V_{\mu\nu} \rangle \omega^{\mu\nu} - \frac{i}{2} b_3 \langle \chi_+ [u_\mu, u_\nu] \rangle \omega^{\mu\nu} + c_1 \langle ([\chi_-, u_\mu] V^\mu) \rangle, \end{aligned} \quad (7.1)$$

where the explicit symmetry breaking is introduced through the χ_\pm , defined in (2.41). The first of the symmetry-breaking terms written out here $\langle \chi_+ \rangle$ produces the pion mass. The next two describe $\rho\omega$ mixing of order p^0 and p^2 . In addition there can be an isospin-violating direct $\omega\pi\pi$ coupling of order p^3 . Under the assumption of resonance saturation in isospin-

violating $\pi\pi$ scattering, we find that the direct $\omega\pi\pi$ coupling is given by (*cf.* (3.13))

$$b_3 = g_1 b_2. \quad (7.2)$$

The final term in (7.1) introduces a chiral-symmetry breaking $\rho\pi\pi$ coupling of order p .

By making appropriate changes of variable, analogous to those used in the context of $\gamma\rho$ mixing [2,3,38], one can diagonalise either the kinetic or mass terms of the Lagrangian to leave it in a form with only one type of mixing. One can also go further and diagonalise the whole free-field part of the vector-meson Lagrangian, in which case all of the isospin violation would appear in the couplings of those mesons. (All of these procedures would produce fields with rather complicated properties under chiral transformations.) Alternatively the theory could be converted into hidden-gauge form, introducing an $\mathcal{O}(p)$ $\omega\pi\pi$ coupling from the change of variables in the mass-type mixing term. From all this, it should be obvious that one can change the strengths of the various mixing terms and isospin-violating couplings in the Lagrangian simply by changing field variables. In calculating any symmetry-breaking amplitude, it is thus not sufficient to know the strength of the mixing and its momentum dependence, one must also have the corresponding symmetry-breaking couplings of the vector mesons [60,61]. This point has recently been stressed by Cohen and Miller [62].

Within the WCCWZ formalism one could set both the momentum-dependent mixing and the direct isospin violating couplings to zero. This would then realise the traditional picture of charge symmetry violation arising purely from a momentum-independent mixing, as suggested in [62]. Note that this refers only to the tree-level mixing parameters and at this level both ρ and ω have zero width. The large width of the ρ arises from its strong coupling to two pions and several authors have pointed out that this can lead to a significant momentum dependence of the $\rho\omega$ mixing [63,64]. This suggests that the inclusion of pion loops is essential in calculations of charge-symmetry breaking effects.

VIII. DISCUSSION

As I have described here, any effective theory of spin-1 mesons and pions can be expressed in WCCWZ, hidden-gauge, massive Yang-Mills or tensor-field form. These schemes correspond to different choices of fields for the spin-1 mesons. The rules for transforming a theory from one form to another have been described in detail, at least for the schemes based on vector fields. Since they are all equivalent, the choice between them must depend on the convenience of the corresponding Lagrangians for a specific calculation. In discussing a particular Lagrangian, we need to be careful to distinguish general features of the formalism used to express it from the properties of the specific coupling constants it contains.

The massive Yang-Mills scheme is rather different from the other three since it is based on the linear realisation of chiral symmetry. Although, correctly treated, it satisfies chiral low-energy theorems, this is not immediately obvious from the Lagrangian since large contributions need to be calculated with delicate cancellations between them. This means that great care needs to be taken if any approximations are made in this approach. In contrast, the other schemes use the nonlinear realisation of chiral symmetry and so the low-energy theorems are built into the forms of the terms appearing in their Lagrangians.

The WCCWZ approach, based on vector fields that transform homogeneously under the nonlinear realisation, imposes no preconceptions about the types of couplings. This makes it particularly useful for comparing theories that have been expressed in different formalisms. In addition, we have seen that the assumptions underlying any particular theory can conveniently be expressed in terms of relations between the coupling constants that appear in its WCCWZ equivalent.

The hidden-gauge scheme uses fields that transform inhomogeneously under the nonlinear realisation and as a result involves couplings that respect a gauge invariance. This makes a convenient formalism to use in the context of vector dominance.

Finally there is the tensor-field formalism. This allows for a rather compact form for the Lagrangian if resonance saturation is assumed. Otherwise it is rather similar to the

WCCWZ approach and does not seem to have any great advantage over that scheme.

By rewriting various theories in WCCWZ form, we have found various relationships amongst their coupling constants. These can be organised hierarchically, according to the underlying physical assumptions that lead to them. First there are the inequalities like (3.6) for the four-point couplings that follow from general principles such as existence of a vacuum or unitarity. Under the assumption of resonance saturation by the ρ and a_1 , these become equalities fixing the values of the four-point couplings. Similarly resonance saturation combined with QCD expectations for the behaviour of current-current correlators fixes the values of certain electromagnetic couplings (3.13,14). The values for the couplings obtained from resonance saturation agree well with those determined from ChPT. Finally there are relations that arise from the phenomenologically successful assumptions of vector dominance in the pion form factor (3.16) and (4.28), universal coupling of the ρ meson (4.24) and the on-shell KSFR relation (4.27). Despite their successes, the underlying origins of these last relations and of resonance saturation remain obscure.

An interesting feature of the hidden-gauge approach is that it involves a parameter g , the gauge coupling for the local symmetry, that has no counterpart in the equivalent WCCWZ Lagrangian. This coupling can be viewed as a parameter in the hidden-gauge vector field. The conventional choice of this field, implicitly used in all applications of the approach, is the one that removes the $\mathcal{O}(p^3)$ part of the $\rho\pi\pi$ coupling. If resonance saturation holds, this choice also removes the $\mathcal{O}(p^2)$ $\gamma\rho$ mixing. Hence this formalism embodies rather naturally the observation that the on-shell ρ couplings satisfy the KSFR relation (2.16).

In summary: the various formalisms for including spin-1 mesons in effective chiral Lagrangians are equivalent. Hence any particular theory could be expressed in terms of the fields of any desired scheme. The choice of scheme is thus a question of convenience for a particular problem. If one is interested in elucidating the physical assumptions built into some given Lagrangian, then conversion of that theory into its WCCWZ equivalent is recommended.

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